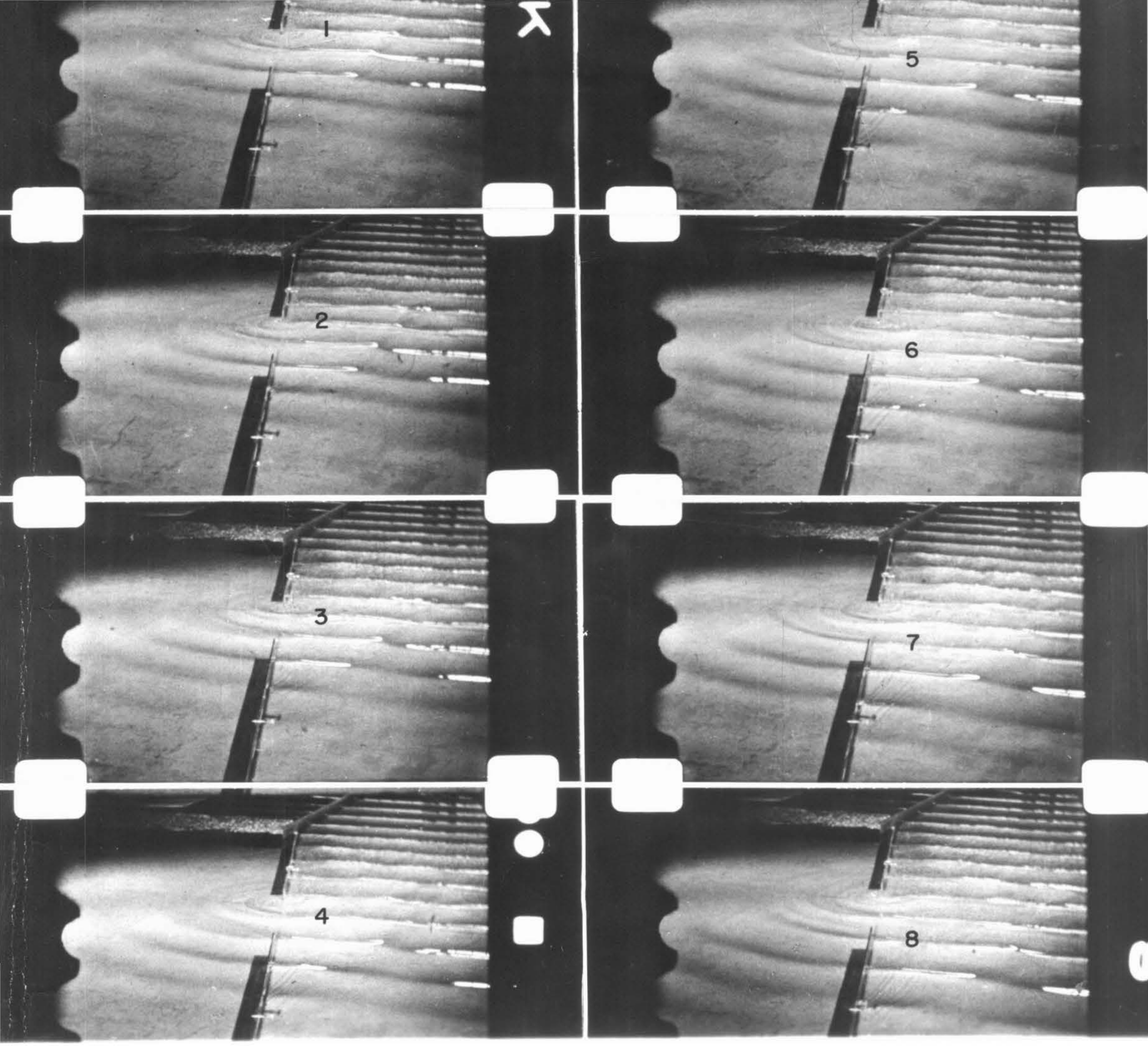

HARBOR DEVELOPMENT STUDY

Theoretical Studies June, 1950

CALIFORNIA INSTITUTE OF TECHNOLOGY
Hydrodynamics Laboratories, Hydraulic Structures Division

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Theoretical Studies - June 1950

GENERAL HARBOR STUDY

Hydrodynamics Laboratories
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The Cover

The cover photograph is an enlargement of 8 frames of a 16-mm motion picture record of the behavior of waves entering a two-wave length gap in a vertical face breakwater at an angle of incidence of 30°.

The sequence follows the complete history of one wave crest, from a position immediately before it reached the seaward end of the opening to a position where the crest was intercepted by the shore leg of the breakwater.

These photographs illustrate one aspect of the phenomenon of diffraction which is the important physical process controlling the wave behavior in this typical situation, the curving of the wave fronts into circular arcs in the lee of the breakwater being an essential characteristic of the diffraction process.

SUMMARY OF THEORETICAL INVESTIGATION

JUNE 1950

I. INTRODUCTION

Investigation of the theoretical aspects of water-wave diffraction through an opening in a breakwater has progressed in two principal directions. First, the work of Penney and Price[1], in adapting Sommerfeld's solution of the optical diffraction problem to the case of water-wave diffraction, has been studied with a view toward generalizing it to suit any angle of wave incidence. In the second place, pursuing the method of a group at the Massachusetts Institute of Technology[2,3,4,5], an application of Mathieu functions to the case of water-wave diffraction has been made, to achieve an exact solution of the problem.

This second phase of the investigation seems to be especially fruitful in that the total amount of energy entering a harbor, and the distribution of energy within the harbor, can be numerically computed. Tables of constants involved in this computation are being published by the National Bureau of Standards, and will be generally available soon.

Computation forms have been devised which permit routine calculation of the desired quantities by sub-professional personnel. Althouth the work of calculation has been reduced to a straightforward task, the amount of work which will be required for a complete set of energy distribution plots, covering a wide range of gap widths and wave directions, is so great that efforts are being made to have this computing performed by IBM machine at the National Bureau of Standards' Institute for Numerical Analysis.

II. AN APPROXIMATE SOLUTION

The cases of water-wave diffraction which were studied by Penney and Price in connection with the Sommerfeld solution, and which are of specific interest to the study at hand, are the following: (1) Diffraction around the end of a rigid breakwater of waves incident normally and obliquely. (2) Transmission of waves approaching at normal incidence through a single gap in a rigid breakwater.

The amplitude and phase conditions resulting from the diffraction of the waves by the gap, as represented by applying the Sommerfeld solution, are of great academic interest. However, because the picture presented is known to be only approximate, as will be shown presently, the method was deemed to be unsatisfactory for the present purpose.

More recently, Putnam and Arthur[6] experimentally checked the theory of Penney and Price for various angles of incidence, using a wave splitter and a wave absorber on the leading face of the single-wing breakwater. Blue and Johnson[7] followed with an experimental study of the diffraction by a gap in a rigid breakwater, with and without a wave-splitter, at normal incidence only. Amplitudes of the diffracted waves were measured at several points beyond the breakwater, and on the basis of the experiment Blue and Johnson concluded that the general form of the diffraction theory was verified for gaps of the order of one and one-half wave lengths and greater. In addition, their results appeared not to have been greatly affected by symmetrical inclination of the barrier wings at angles as low as 45° to the line of symmetry.

A rigid barrier is considered to be one at which the normal component of fluid velocity is zero, $\frac{\partial \phi}{\partial n} = 0$, where ϕ is the velocity potential and n is the direction of the normal to the barrier. An ideal cushion-type breakwater is one at which the pressure remains constant, implying that $\frac{\partial \phi}{\partial t} = 0$; in this case the vertical amplitude of water motion is zero, and the breakwater boundary moves horizontally with the water. The wave absorber used by Putnam and Arthur, assuming it was not a perfect cushion, was something between the two ideals, and therefore Sommerfeld's rigid-barrier condition was not satisfied. In the experiments at this laboratory an attempt is being made to represent as nearly as possible in the model basin the conditions existing in an actual prototype case. For that reason a rigid barrier is employed, since it is considered to be more realistic. Wave-splitters, too, are omitted, in order that any effect which the wave reflected from the front face of the barrier may have on the diffraction process may have an opportunity to assert itself, especially at angles of incidence other than normal. Only straight breakwater alignments have been used so far, at angles of incidence of from 0° to 90° in relatively small increments. It is expected that in the future other alignments will be studied in connection with theoretical considerations to be discussed later.

The application of Sommerfeld's solution to the water-wave diffraction situation provides, for the case of the single-wing or semi-infinite breakwater, an exact method of predicting wave phase and amplitude conditions at different points affected by reflection and diffraction. It is assumed that the motion is irrotational, at any fixed boundary the normal component of velocity is zero, at the free surface the pressure is constant, the water is of uniform depth, and the vertical component of water motion is infinitesimal.

Since the fluid motion is considered to be irrotational, a velocity potential exists and satisfies the Laplace equation:

$$\text{(a)} \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0.$$

With the assumptions which have been made the elevation of the water surface is shown to be of a form

$$\text{(b)} \qquad \eta = \frac{ikc}{g} \; e^{ikct} \cosh kd \cdot F\,(x,y),$$

where

$$\text{(c)} \qquad \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + k^2 F = 0.$$

Equation (b) resembles the solution given by Lamb[13] for the identical situation. The only factor which is affected by diffraction is F (x,y), and for that reason attention is focused on it to study the diffraction behavior. For incident waves traveling in the positive y-direction,

$$\text{(d)} \qquad F\,(x,y) = e^{-iky}.$$

Penney and Price show that conditions existing in this water-diffraction case are identical with those satisfied by Sommerfeld's solution for the light-diffraction problem. The Sommerfeld solution is, in rectangular coordinates,

$$\text{(e)} \quad F\,(x,y) = \frac{1+i}{2} \left\{ e^{-iky} \int_{-\infty}^{\sigma} e^{-\frac{1}{2}\pi i u^2}\, du + e^{iky} \int_{-\infty}^{\sigma'} e^{-\frac{1}{2}\pi i u^2}\, du \right\}$$

where $\quad \sigma = \sqrt{\frac{4}{\lambda}(r - y)}, \quad \sigma' = \sqrt{\frac{4}{\lambda}(r + y)}, \quad \lambda = \frac{2\pi}{k}, \quad r = \sqrt{x^2 + y^2}$

The signs for σ and σ' depend upon the position of point (x,y) in the $x - y$ plane. Therefore, for each of the three regions examined by this method, the unaffected, the sheltered and the reflection regions, various combinations of the incident wave and the reflection and diffraction components, $g\ (r,\ y)$ and $f\ (r,\ y)$, arise.

The Sommerfeld equation proves to be of a convenient form, because it lends itself quite readily to solution by tabulated values of Fresnel's integrals or by means of Cornu's Spiral. The argument and modulus of $F(x,y)$ so determined for any specific point in one of the three regions, compared with that of $F\ (x,\ y) = e^{-iky}$ for the incident waves, reveal what happens to the phase and amplitude respectively during the process of reflection and diffraction.

A study of the complete wave picture in the three regions reveals some interesting facts. At the edge of the geometric shadow, passing into the sheltered region, the wave height is reduced to one-half that of the incident waves, and there is a $\lambda/8$ lag in phase in the diffracted component. In this region the diffracted waves appear very nearly as arcs of circles, concentric about the end of the breakwater. This wave amplitude along a given crest is progressively reduced as the barrier is approached, the rate of decay being a function of r. It is seen that points having a given amplitude reduction factor $\frac{1}{n}$ lie along a parabola. Fig. 3 illustrates the notation used and some results of this method.

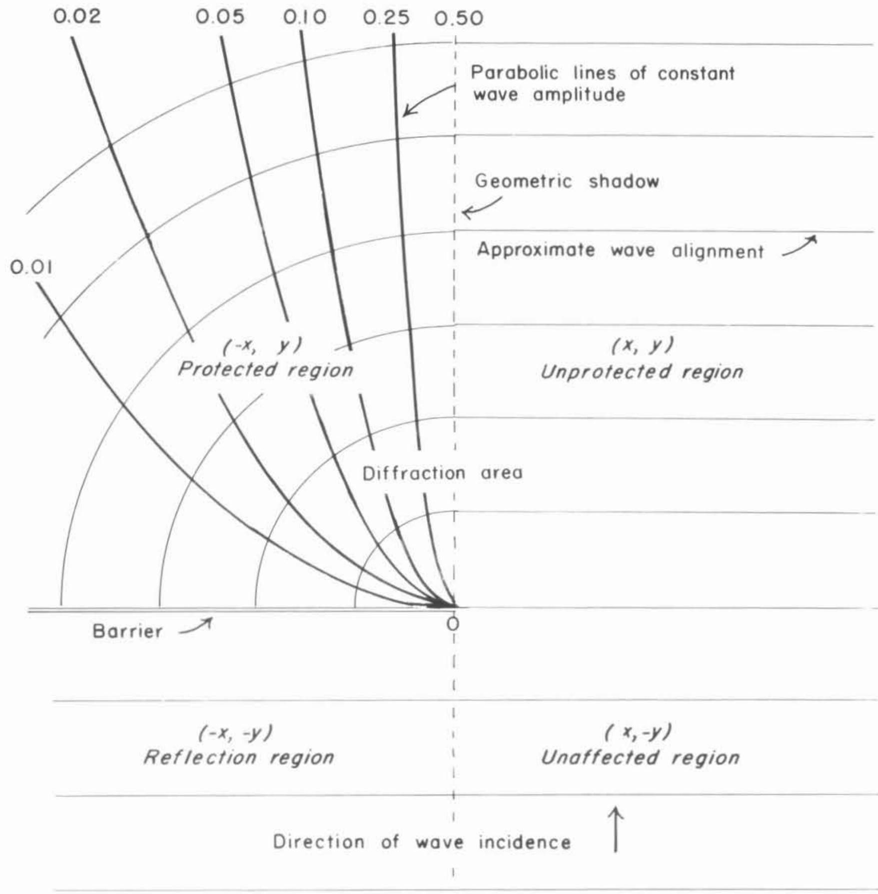


Fig. 1 - Notation and results for Penney and Price diffraction problem

Proceeding from the edge of the shadow into the unprotected region, the diffracted component shifts phase by $\lambda/2$, and the amplitude of the diffracted wave approaches zero at points along the continuation of the barrier. Reinforcement occurs in such a way that the amplitude of the resultant along a line just beyond the edge of the shadow is greater than that of the incident wave. At the boundary between the reflection zone and the open water, diffraction of the reflected wave occurs in a manner symmetric with respect to that just described.

The method of Penney and Price for a semi-breakwater was seen to be an exact solution in that exact boundary conditions were imposed on the water motion. The theory was extended to the case of breakwater with a gap by superposing the solution obtained for a given point by using a single-wing to the left of the origin, upon that resulting from the right wing, the origin being taken at the center of the gap. It will be recalled that in the case of a rigid barrier, the absolute value of the normal gradient of $F(x, y)$ must be zero at the barrier. For the superposed solution, the same requirement must be satisfied; the modulus of the gradient of the complex function must disappear at points along the breakwater. The yardstick by which the exactness of this superposed solution is measured, then, is how closely the modulus of the normal gradient of the combined function approaches zero along the barrier. Penney and Price show that when the gap is greater than two wave lengths, this modulus at the end of one wing never exceeds about 16% of the modulus of the incident wave's rate of decay. As the gap size approaches zero the modulus of the gradient becomes infinite, because of the fact that an asymptotic solution is involved which is not valid for small values of r. The method was extended at this Laboratory to include angles of incidence other than the normal case developed by Penney and Price, to make it applicable to as general a case as possible. The relationship which was developed reduced to the same gradient error of about 16% for a gap of two wave lengths and a normal angle of approach.

The fact that a certain amount of gradient error is present along the boundary is not, however, any assurance that the solution at any point in the water is even that accurate. That factor, along with the complete unsuitability for small gaps, makes the method of Penney and Price seem weak as a basis for correlation with experimental data, and for precise prediction of wave behavior. Other modes of attack, therefore, were investigated in the search for an exact solution.

III. AN EXACT SOLUTION

The unsatisfactory features of the Sommerfeld solution are overcome and an exact solution obtained by separating the wave equation in elliptic cylinder coordinates and computing the resulting Mathieu functions, a method which has been satisfactorily applied to the diffraction of electromagnetic and sound waves. It is believed that the current work represents the first application of Mathieu functions to the diffraction of water waves. So far the results appear promising; the resulting solution is exact for any angle of incidence and for any breakwater gap, even one which approaches zero. The total amount of energy entering the harbor may be computed and reduced to a directional distribution at a distance several wave lengths from the gap.

Mathieu functions have received the attention of mathematicians over a period of the past century. For a few of the relatively recent discussions of Mathieu functions see references 8 to 11. It has been said that among mathematical physicists Mathieu functions rank closely after Bessel, Legendre and related functions in an order of priority of numerical tabulation. Such tabulations have been made in the past, based on various parameters, but the first one which appears to be directly usable in the diffraction problem is one set up by Stratton, Morse, Chu and Hutner(5) based on relationships which they develop. These investigators and Rubenstein have discussed quite thoroughly the use of elliptic cylinder, as well as prolate and oblate spheroidal, coordinates in relation to wave diffraction. Recently the Institute for Numerical Analysis of the National Bureau of Standards has enlarged upon and extended these tables(11) with a slight change in notation, so that there is now available a store of practical material on the Mathieu functions.

The problem is set up in the specified coordinate system by means of a conformal transformation of the form:

$$\text{(f)} \qquad x + iy = {}^{d}/_{2} \cosh (\xi + i\phi),$$

or

$$\text{(g)} \qquad \begin{cases} x = {}^{d}/_{2} \cosh \xi \cos \phi \\ y = {}^{d}/_{2} \sinh \xi \sin \phi \\ z = z \end{cases}$$

The new set of coordinate surfaces which arises may be shown to consist of confocal elliptic and hyperbolic cylinders, and planes perpendicular to the axes of the cylinders. Since, however, the propagation vector of the wave is taken in the x - y plane, the z coordinate may be eliminated from consideration, and the lines of constant ξ and ϕ become, respectively, confocal ellipses and hyperbolas of focal length d. The degenerate form in each case is a straight line; for $\xi = 0$ the ellipses degenerate into a straight line of length d, and the hyperbolas become, for $\phi = 0$, a straight line with a gap of width d. The former could be used in considering diffraction around both ends of a barrier of finite length. It is the line with a gap which is of interest to this Laboratory, however, as it represents a breakwater with an opening.

The two-dimensional wave equation is

$$\text{(h)} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2},$$

where v is the wave velocity. Substituting the elliptic cylinder coordinates into this equation, the variables ϕ, ξ and time may be separated in the standard manner, and the following differential equations result:

$$\text{(i)} \qquad \frac{d^2H}{d\phi^2} + (b - s\cos^2\phi)\,H = 0$$

$$\frac{d^2G}{d\xi^2} + (s\cosh^2\xi - b)\,G = 0$$

where $s = (\frac{\pi d}{\lambda})^2$, and H and G are functions of only ϕ and ξ, respectively. Solutions of these equations are, of course, solutions of the wave equation from which they arise. The first of these is known as Mathieu's equation and the second as the modified Mathieu equation, since it may be obtained from the first by substituting $\phi = i\xi$. First order even and odd solutions of Mathieu equation are usually represented by $Se_r(s,\phi)$ and $So_r(s,\phi)$, corresponding to $H(\phi)$. First order solutions of the modified equation are $Je_r(s,\xi)$ and $Jo_r(s,\xi)$, corresponding to the function $G(\xi)$.

Taking an infinite number of values of the parameter b, of course, results in an infinite number of solutions of the differential equations for a given width of gap. Only for a certain set of characteristic values of b are the solutions periodic, however, and of this set a relatively small number of values of b give rise to solutions which are of period π or 2π. It is these particular periodic solutions which are commonly referred to as Mathieu functions. Thus, the first order solutions of the Mathieu equation, or the first angular functions, are

$$\text{(j)} \qquad \begin{aligned} Se_r(s,\phi) &= \sum_{k=0}^{\infty}{}' De_k \cos k\phi \\ So_r(s,\phi) &= \sum_{k=0}^{\infty}{}' Do_k \sin k\phi \end{aligned}$$

The first and second order solutions of the modified equation are related to each other and to the first order angular solutions by factors of proportionality, or joining factors, which are readily computable.

The subscripts r in the solutions above are index numbers, corresponding to the increasing individual values of the parameter b yielding the proper periodic functions. Ordinarily, in practical computing work, it is necessary to use only values of r from zero to about four or five for reasons to be seen later. The primes indicate that for even values of r (0, 2, 4 . . .), only even values of k are included in the summation, and for odd r (1, 3, 5 . . .) only odd values of k are summed.

The Mathieu coefficients De_k, for example, may be computed by substituting the first of equations (j) into the Mathieu equation, using the series representation for trigonometric functions, and equating coefficients of equal powers of ϕ in the resulting equation. The Mathieu coefficients are then seen to satisfy certain recursion relationships which may be represented by a continued fraction whose value may be computed, provided the value of the first coefficient is known. This first coefficient, De_o or De_1, is effectively established by choosing:

$$
\text{(k)} \quad \begin{cases} Se_m\,(s,\ 0) = 1 \\ \left[\left(\frac{d}{\partial\phi}\right) Se_m\,(s,\ \phi) \right]_{\phi = 0} = 0 \end{cases}
$$

This choice also imposes the desired boundary conditions along the rigid double-wing barrier which is of present interest.

Stratton et al[5] tabulated values of the even and odd coefficients with the associated characteristic values of b corresponding to r's of from zero to four, and for values of c ($= \sqrt{s}$) up to 4.5. This tabulation corresponds to gate widths roughly up to $1\frac{1}{2}\lambda$, but with greater intervals between values than is most desirable for practical application. Based on these tables and applicable formulae given by the Massachusetts Institute of Technology group, calculation sheets were made up for use at the Hydraulic Structures Laboratory by secretarial labor, to determine the total energy transmitted through the gap, and the directional distribution of energy within the harbor.

Use of the sheets proved to be quite laborious, in that much of the calculation involved the determination of a phase angle δ represented by the relationship

$$\text{(1)} \qquad \operatorname{ctn} \delta_r = 2\pi \mu_r \lambda_r,$$

where μ_r and λ_r are proportionality factors relating the first order and second order solutions. This difficulty has been largely overcome, however, by the use of preliminary copies of tables which have been computed by the National Bureau of Standards, and which are now being published. In these tables, values of new joining factors, $g_{e,r}$, $g_{o,r}$, $f_{e,r}$, and $f_{o,r}$, are listed for values of r from zero to fifteen, and for values of s which carry far beyond any gate widths which are at present anticipated. For the rigid barrier, double-wing situation, the constant $f_{e,r}$ is the value of $\operatorname{ctn} \delta_r$ which previously had to be computed. It is the rapidity with which δ approaches zero that determines the range of values of r which must be considered. At the same time, characteristic values and even and odd Mathieu coefficients for this greatly extended range were included. Based on these tables, new computation sheets were developed to determine the specific values needed in the laboratory program; these tables reduce the computational labor to about one-half of that previously required. Despite this advantage, the volume of computations required for a complete set of curves is beyond the resources of this Laboratory, hence efforts are being made to enlist the aid of the Institute for Numerical Analysis in this important phase of the study.

Morse[4] demonstrates the expansion of a plane wave in a series of Mathieu functions, and from the consideration of zero normal gradient at the barrier, develops the solution of the wave equation in terms of the plane wave plus the scattered wave. Morse and Rubenstein[2] pursue these relationships further and obtain the value of an intensity factor, which is the ratio of the energy transmitted in a given direction, ϕ, to the energy of waves incident at an angle u. By virtue of the fact that the energy is proportional to the square of the wave amplitude, a series of cross products form the expression for the intensity factor:

$$(m) \qquad I = \frac{4\pi}{\sqrt{s}} \sum_{m,n} \frac{1}{Nm\ Nn} \sin\gamma m \ \sin\gamma n \ \cos(\gamma n - \gamma m)$$

$$\times\ Se_m(s,\ u)\ Se_n(s,\ u)\ Se_m(s,\ \phi) \times Se_n(s,\ \phi),$$

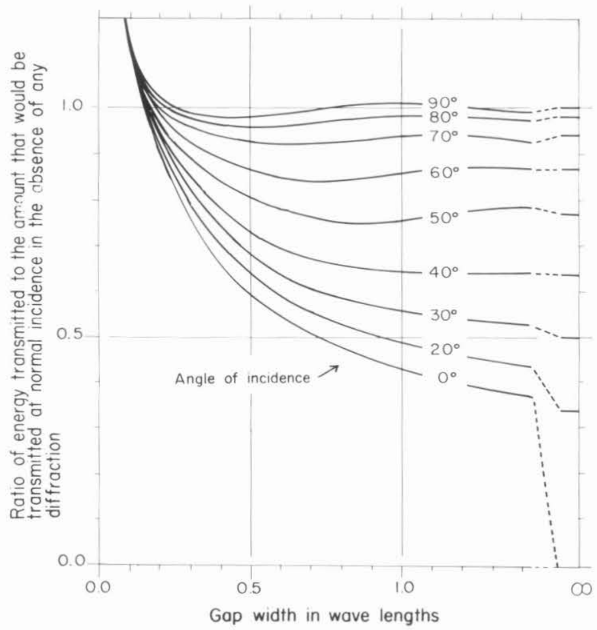
where γ is the phase angle mentioned previously. To put this equation to practical use, it is further noted that the intensity scattered at an angle ϕ at a sufficient distance R from the center of the gap is $({}^{d}/R)$ I. By "sufficient distance" it is implied that R is large enough that the asymptotic relationships developed by Morse and Rubenstein are valid.

If I x d is integrated over ϕ from 0 to π, there is obtained the total transmission factor:

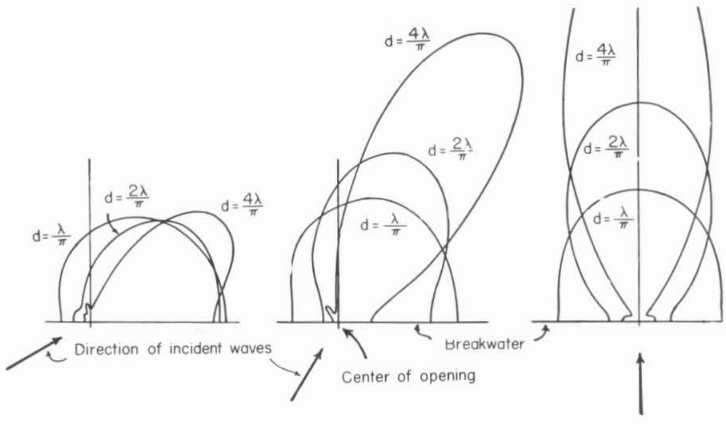
$$\text{(n)} \qquad T = \frac{2\pi}{\sqrt{s}} \sum_m \frac{1}{N_m} \sin^2 \gamma_m \left[Se_m \ (s, \ u) \right]^2$$

The total transmission factor may be defined as the ratio of the amount of energy actually transmitted by the gap to the amount of energy which geometrical optics predicts would be transmitted at normal incidence. Some results of this method, as presented by Morse and Rubenstein[2] are shown in Fig. 2.

It is expected that this theory of wave diffraction will be verified by experimental data which has been recorded but not yet completely evaluated. Comparison of intensity factors and total transmission factors, determined theoretically and experimentally, should reveal a close correlation, and if any unforeseen discrepancies in either the application of this theory or in the model set-up appear, the required modifications may then be effected.



(a) Total energy transmission as a function of gap width



(b) Polar diagrams of wave energy distribution

Fig. 2 - Theoretical results due to Morse and Rubenstein

IV. FUTURE STUDIES

In line for future investigation are various other aspects of the problem of predicting the wave behavior within harbor areas. While the exact method just outlined seems to be satisfactory for a straight breakwater, it is now desirable to concentrate on the condition of breakwaters having wings inclined to each other, in order to produce results which are as generally applicable as possible. Elliptic cylinder coordinates, of course, cannot be employed, since the degenerate hyperbola is a straight line with a gap. It is possible to construct coordinates which degenerate to two lines inclined to each other at some angle of between 0° and 180°; however, this set of coordinates is not one of the two-dimensional systems for which the wave equation is separable, hence it is not adaptable for the present purpose.

A different method of attack which may lend itself to this general case is the variational principle for diffracted wave amplitude. This approach has been examined by Levine and Schwinger[12] in the case of plane waves being diffracted by an aperture in an infinite plane screen. The wave function at an arbitrary point in space is expressed in terms of its value in the aperture, and constructed so as to satisfy the prescribed boundary conditions. An integral equation is involved which may prove to be unmanageable in the present application. However, if such an approach does prove successful it is believed that much progress will have been made in the generalization of the water-wave diffraction case.

Also, it is desired to investigate other solutions for certain limited cases which may come to attention. For example Lamb[13] shows that in the case of diffraction of a normally-incident wave through a slot in a plane screen, when the width of the opening is very small compared to the wave length, the ratio of energy in the transmitted wave to that in the primary waves is

$$R_E = \frac{\frac{\pi^2}{4}}{kb \left\{ \left(\log \frac{kb}{4} + \gamma\right)^2 + \frac{\pi^2}{4} \right\}} \times 2b \qquad \text{(o)}$$

Any such special solution which comes to light will be checked numerically against the most general solution available at the moment for closeness of agreement.

Finally, it is desirable, of course, to maintain a constant search for new modes of attack which have previously been applied to water wave diffraction, or to any other kind of diffraction. In the light of what has been accomplished to date, it would seem that mathematical prediction of harbor wave behavior for quite general cases is highly possible.

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